Conditions of invariance and extinction for stochastic model of control population

L.I. Rodina

Udmurt State University, Universitetskaya 1, Izhevsk 426034, Russia
e-mail: box0589@udmnet.ru

Abstract.

New probabilistic model of population dynamics is analyzed. Conditions of invariance and conditions of asymptotic extinction with probability one for population were determined. Results of analysis are illustrated by an example of bisexual population dynamics.

Keywords: stochastic model of population dynamics, probability of population extinction

Introduction

This publication is a continuation of our previous publications (Rodina 2013, 2014) which are devoted to new stochastic model of population dynamics. Considering model is generalization of models described in publications by L.V. Nedorezov (1997) and L.V. Nedorezov and Yu.V. Utyupin (2003, 2011), and based on the theory of differential equations with random coefficients and differential equations with impulses.

Within the framework of deterministic mathematical models of isolated population dynamics it is assumed that the death of individuals has continuous character, and appearance of individuals of new generations takes place in some fixed time moments τ_k. For considering model we assume that population development on time intervals (τ_k, τ_{k+1}), as well as the moments τ_k are depend on various changes of an environment, and therefore dynamics of population are described by system with random coefficients.

Let's designate through $\mathfrak{M}(\sigma)$ the set which limits the size of population and depends on random parameter σ. For considering hypotetical population we investigate conditions of invariance and weakly invariance of set $\mathfrak{M}(\sigma)$ concerning control system. Set $\mathfrak{M}(\sigma)$ is called invariant if for any initial point $(t_0, x_0) \in \mathfrak{M}(\sigma)$ each solution $\varphi(t, \sigma, x_0)$ of system with initial condition $\varphi(t_0, \sigma, x_0) = x_0$ satisfies to inclusion $(t, \varphi(t, \sigma, x_0)) \in \mathfrak{M}(\sigma)$ for all $t \geqslant t_0$. If $\mathfrak{M}(\sigma)$ does not possess the property of invariance, but there is an admissible control under which the solution $\varphi(t,\sigma,x_0)$ all time is limited by this set, then the set $\mathfrak{M}(\sigma)$ is called weakly invariant. For various probabilistic models a situation is often observed when property of invariance is satisfied not for all, but for almost all values of parameter σ. In such conditions we say that the set $\mathfrak{M}(\sigma)$ is invariant with probability one.

Let's notice, that for various problems the set $\mathfrak{M}(\sigma)$ can limit the size of population from above or from below or may have more difficult structure. For example, if we investigate the population of close-living species of forest insects then it is expedient to consider the set limiting the size of population from above and to choose such controls which are capable to prevent mass flashes of quantity of insects. Now let's consider the population which is subject to a craft, when the moments of trade preparations and the sizes of these preparations are random variables. The following problem is of interest for such model: to find conditions of existence of control directed on increase of the population and its preservation at certain level. Thus, for given population we will choose set $\mathfrak{M}(\sigma)$, limiting its size from below.

We determined also conditions of asymptotical extinction and conditions for control leading population to extinction. Results of current publication can find application in practical problems conserning control influences which are directed on increase of population (preservation of rare types of animals, increase of the size of exploited population), or on its reduction (control of the number of harmful insects, epidemiology problems etc.).

Deterministic and stochastic discrete-continuous models of population dynamics

In various models of population dynamics (for example, models with phase structure, age structure, gender structure, models describing dynamics of individuals of different types and various age classes) it is assumed that transition from one type or class of individuals into another one has "jump character", and it is carried out in fixed time moments τ_k. These models with continuous-discrete behavior of trajectories are described by systems of differential equations with impulses (examples can be found in following publications: Nedorezov, 1997; Nedorezov, Utyupin, 2003, 2011; Iannelli, Martcheva, Milner, 2005). Let's consider a model of population dynamics describing by following control system

$$\begin{aligned} &\dot{x} = f(t,x,u), \quad t \neq \tau_k, \\ &\Delta x\big|_{t=\tau_k} = g(x,w), \quad (t,x,u,w) \in \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^p, \end{aligned} \tag{1}$$

where $\tau_k = kT$, $k = 1,2,\ldots,$ $\mathbb{R}^n$ is a standard Euclidean n-dimension space with scalar product $\langle x,y\rangle$ and norm $|x| = \sqrt{\langle x,x\rangle}$. Admissible controls $u(t)$ are limited measurable functions with values in compact set $U \subset \mathbb{R}^m$, w is control vector influencing behavior of system in time moments τ_k and accepting values in compact set $W \subset \mathbb{R}^p$. We assume that functions $f(t,x,u)$ and $g(x,w)$ are continuous on set of variables and solutions of system (1) are continuous at the left. We will also use the next designations: $\varrho(x,M) \doteq \inf_{y \in M} |x-y|$ is a distance from a point $x \in \mathbb{R}^n$ to the set M in $\mathbb{R}^n$, $\mathrm{fr}M$ is a border, $\mathrm{int}M$ is an interior of the set M.

The full description of stochastic discrete-continuous model one can find in our publications (Rodina 2013, 2014). Here we will result its short description as it is necessary for the further statements. It is natural to suppose that population size changing on intervals (τ_k, τ_{k+1}), as well as moments τ_k, are determined by various natural conditions. Therefore we have to assume that on each time interval (τ_k, τ_{k+1}) function f depends on random parameter ψ_k, accepting value in the set Ψ, and lengths of intervals $\theta_k = \tau_k - \tau_{k-1}$, $k = 2, 3, \ldots$ between the moments of appearance of new generations are random variables with distribution function $G(t)$. We assume that this distribution is concentrated on a segment $[\alpha, \beta]$, where $0 < \alpha < \beta < \infty$. In that specific case when all variables ψ_k, θ_k are constants, stochastic model coincides with deterministic one, therefore it is a generalization of deterministic model.

Let's assume that dynamics of population is given by control system

$$\begin{aligned} \dot{x} &= f(h^t\sigma, x, u), \quad t \neq \tau_k(\sigma), \\ \Delta x\big|_{t=\tau_k(\sigma)} &= g(x,w), \quad (t,\sigma,x,u,w) \in \mathbb{R} \times \Sigma \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p, \end{aligned} \tag{2}$$

that generated by metric dynamical system $(\Sigma, \mathfrak{A}, \mu, h^t)$. We assume that function $(x,w) \to g(x,w)$ and function $(t,x,u) \to f(h^t\sigma, x, u)$ for every fixed $\sigma \in \Sigma$ is continuous on set of variables and controls u and w accept values in compact sets $U \subset \mathbb{R}^m$ and $W \subset \mathbb{R}^p$ accordingly.

Let's remind that probability space $(\Sigma, \mathfrak{A}, \mu)$ is direct product of probability spaces $(\Sigma_1, \mathfrak{A}_1, \mu_1)$ and $(\Sigma_2, \mathfrak{A}_2, \mu_2)$. Here Σ_1 is the set of numerical sequences $\theta = (\theta_1, \ldots, \theta_k, \ldots)$, where $\theta_k \in [\alpha, \beta]$, system of sets $\mathfrak{A}_1$ is the least sigma-algebra generated by cylindrical sets

$$E_k \doteq \{\theta \in \Sigma_1 : \theta_1 \in I_1, \ldots, \theta_k \in I_k\},$$

where $I_i \doteq (t_i, s_i]$, and the probabilistic measure μ_1 is defined as follows. For each interval I_i, $i \geqslant 2$, we define a probability measure $\widetilde{\mu}_1(I_i) = G(s_i) - G(t_i)$ by means of distribution function $G(t)$, on algebra of cylindrical sets we construct a measure

$$\widetilde{\mu}_1(E_k) = \widetilde{\mu}_1(I_1)\widetilde{\mu}_1(I_2)\ldots\widetilde{\mu}_1(I_k),$$

then owing to theorem of A.H Kolmogorov (Shiryaev, p.176, 1989) on measurable space $(\Sigma_1, \mathfrak{A}_1)$ there is a unique probabilistic measure μ_1, which is continuation of $\widetilde{\mu}_1$ on sigma-algebra $\mathfrak{A}_1$.

Further, let are fixed set Ψ and the sigma-algebra of its subsets $\mathfrak{A}_0$, where probabilistic measure $\widetilde{\mu}_2$ is defined. Let's designate through Σ_2 the set of sequences

$$\Sigma_2 \doteq \{\varphi : \varphi = (\psi_0, \psi_1, \dots, \psi_k, \dots),\, \psi_k \in \Psi\}.$$

We define a measure $\widetilde{\mu}_2\big(\varphi \in \Sigma_2 : \psi_1 \in \Psi_1, \dots, \psi_k \in \Psi_k\big) = \widetilde{\mu}_2(\Psi_1)\widetilde{\mu}_2(\Psi_2)\dots\widetilde{\mu}_2(\Psi_k)$, where $\Psi_i \in \mathfrak{A}_0$ and a measure μ_2 as measure continuation of $\widetilde{\mu}_2$ on sigma-algebra $\mathfrak{A}_2$. On probability space $(\Sigma, \mathfrak{A}, \mu)$ we define the shift transformation $h^t\sigma$ that keeps a measure $\mu = \mu_1 \times \mu_2$ which is direct product of probability measures μ_1 and μ_2. It means, that $\mu_1 \times \mu_2(A \times B) = \mu_1(A)\mu_2(B)$ for all sets $A \in \mathfrak{A}_1$, $B \in \mathfrak{A}_2$.

The basic definitions

Let's consider stochastic system (2) which describes population dynamics with phase or age structure. We will demand that solutions of system (2) are non-negative at non-negative initial conditions. This requirement is satisfied for system $\dot{x} = f(h^t\sigma, x, u)$ if and only if the function f satisfies to a condition of *quasi-positive* (Kuzenkov, Ryabova, 2007). Let's formulate a similar condition for system with impulse (2). We designate

$$\mathbb{R}^n_+ \doteq \{x \in \mathbb{R}^n : x_1 \geqslant 0, \dots, x_n \geqslant 0\}.$$

Definition 1. We say, that functions $f(h^t\sigma, x, u)$ and $g(x, w)$ satisfy to a *condition of quasi-positive,* if for any $(t, \sigma, x) \in \mathbb{R}_+ \times \Sigma \times \mathbb{R}^n_+$ and any admissible controls take place following inequalities:

$$f_i(h^t\sigma, x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_n, u_1, \dots, u_m) \geqslant 0, \quad i = 1, \dots, n, \tag{3}$$

$$x_i + g_i(x_1, \dots, x_n, w_1, \dots, w_p) \geqslant 0, \quad i = 1, \dots, n. \tag{4}$$

Let $x(t, \sigma, x_0)$ is the solution of system $\dot{x} = f(h^t\sigma, x, u)$, satisfying to the initial condition $x(0, \sigma, x_0) = x_0$. If the inequality (3) is true and $x_0 \in \mathbb{R}^n_+$, then $x(t, \sigma, x_0) \in \mathbb{R}^n_+$ for all $t \geqslant 0$ (Kuzenkov, Ryabova, p. 34, 2007). From the inequality (4) it follows that solutions of system with impulse (2) are non-negative.

Following A. F. Filippov, we will put in conformity to system (2) differential inclusion

$$\dot{x} \in F(h^t\sigma, x), \quad F(\sigma, x) = \operatorname{co} \widetilde{F}(\sigma, x), \tag{5}$$

where for each fixed point $(\sigma, x) \in \Sigma \times \mathbb{R}^n$ set $\widetilde{F}(\sigma, x)$ consists of all limiting values of function $(t, x) \to f\big(h^t\sigma, x, U\big)$ as $(t_i, x_i) \to (0, x)$. We assume, that at fixed (σ, x) the set $F(\sigma, x)$ is convex and compact and for every $\sigma \in \Sigma$ function $(t, x) \to F(h^t\sigma, x)$ is upper semi-continuous. Then for any $\sigma \in \Sigma$ there is a local solution of Cauchy problem $\dot{x} \in F(h^t\sigma, x)$, $x(0) = x_0$ (Blagodatskikh, Filippov, 1985).

Let's designate $\Omega \doteq \Sigma \times \mathrm{comp}(\mathbb{R}^n)$, where $\mathrm{comp}(\mathbb{R}^n)$ is the space of nonempty compact subsets of Euclidean space $\mathbb{R}^n$. To each set $X \in \mathrm{comp}(\mathbb{R}^n)$ and to time moment $t \geqslant 0$ we will put in correspondence the set $D(t, \omega)$, where $\omega = (\sigma, X)$, consisting of all values in the moment t of solutions $t \to \varphi(t, \sigma, x)$ of inclusion (5), when the initial condition $\varphi(0, \sigma, x) = x$ runs all set X. The set $D(t, \omega)$ is called a *set of attainability* of control system (2). We assume, that for given set $X \in \mathrm{comp}(\mathbb{R}^n)$ the set $D(t, \omega)$ exists at all $t \geqslant 0$; it means, that for every $x \in X$ there is a solution $\varphi(t, \sigma, x)$ of differential inclusion (5), satisfying to initial condition $\varphi(0, \sigma, x) = x$ and continued on $\mathbb{R}_+ = [0, \infty)$.

For every $\sigma \in \Sigma$ we consider a continuous mapping $t \to M(h^t\sigma)$ with values in space $\mathrm{comp}(\mathbb{R}^n)$ and set $\mathfrak{M}(\sigma) = \{(t, x) : t \geqslant 0,\, x \in M(h^t\sigma)\}$. Let positive number r is fixed. We designate through $O_r(x_0) = \{x \in \mathbb{R}^n : |x - x_0| \leqslant r\}$ closed sphere of radius r with the centre in a point $x_0 \in \mathbb{R}^n$, through $M^r(\sigma) = M(\sigma) + O_r(0)$ closed r-neighbourhood of $M(\sigma)$ in $\mathbb{R}^n$, through $N^r(\sigma) = M^r(\sigma) \setminus M(\sigma)$ external r-neighbourhood of boundary of $M(\sigma)$, also we will construct the set $\mathfrak{N}^r(\sigma) = \{(t, x) : t \geqslant 0,\, x \in N^r(h^t\sigma)\}$.

Definition 2 (Panasenko, Tonkov, 2009). Scalar function $x \to V(\sigma, x)$ is called *Lyapunov's function* (concerning the set $\mathfrak{M}(\sigma)$) if function $(t, x) \to V(h^t\sigma, x)$ satisfies to local Lipschitz condition and to following conditions:

1) $V(h^t\sigma, x) = 0$ for all $(t, x) \in \mathfrak{M}(\sigma)$;
2) $V(h^t\sigma, x) > 0$ for some $r > 0$ and all $(t, x) \in \mathfrak{N}^r(\sigma)$.

Definition 3. Lyapunov's function $x \to V(\sigma, x)$ is called *definitely positive* (on set $\mathfrak{M}^r(\sigma)$), if for every $\varepsilon \in (0, r)$ will be such $\delta > 0$, that $V(h^t\sigma, x) > \delta$ for all

$$(t, x) \notin \mathfrak{M}^{\varepsilon}(\sigma) \doteq \{(t, x) : t \geqslant 0,\, x \in M^{\varepsilon}(h^t\sigma)\}.$$

Definition 4. For locally Lipschitz functions $V(\sigma, x)$ *generalized derivative* in a point $(\sigma, x) \in \Sigma \times \mathbb{R}^n$ in a vector direction $d \in \mathbb{R}^n$ (F.Clark's derivative) is called the following limit (Clarke, p. 17, 1983):

$$V^o(\sigma, x; d) \doteq \limsup_{(\vartheta, y, \varepsilon) \to (\sigma, x, +0)} \frac{V(h^{\varepsilon}\vartheta, y + \varepsilon d) - V(\vartheta, y)}{\varepsilon},$$

and expressions $V^o_{\min}(\sigma,x) \doteq \inf\limits_{d\in F(\sigma,x)} V^o(\sigma,x;d)$, $V^o_{\max}(\sigma,x) \doteq \sup\limits_{d\in F(\sigma,x)} V^o(\sigma,x;d)$ are called *lower* and *upper derivatives* of functions V owing to differential inclusion (5).

Conditions of invariance of set concerning control system

In this section we investigate conditions of invariance and weakly invariance of given set $\mathfrak{M}(\sigma)$ concerning control system (2). Let's consider differential equation with impulse

$$\begin{aligned} \dot{z} = q(h^t\sigma, z), \quad t \neq \tau_k(\sigma), \\ \Delta z\big|_{t=\tau_k(\sigma)} = l(z), \quad (t,\sigma,z) \in \mathbb{R}\times\Sigma\times\mathbb{R}, \end{aligned} \tag{6}$$

that generated by metric dynamical system $(\Sigma, \mathfrak{A}, \mu, h^t)$. We assume that solutions of equation (6) are continuous at the left; $q(\sigma,0)\equiv 0$ and the equation (6) possess the property of uniqueness of Cauchy problem. We consider the function $L(z) \doteq l(z) + z$ and assume that this function is increasing, $L(0)=0$ and $L(z)\geqslant 0$ if $z>0$.

Definition 5 (Panasenko, Tonkov, 2008, 2009). The set $\mathfrak{M}(\sigma)$ is called *positively invariant* (concerning system (2)) if for any initial point $(t_0,x_0)\in\mathfrak{M}(\sigma)$ each solution $\varphi(t,\sigma,x_0)$ of system (2) with the initial condition $\varphi(t_0,\sigma,x_0)=x_0$ satisfies to inclusion $(t,\varphi(t,\sigma,x_0))\in\mathfrak{M}(\sigma)$ for all $t\geqslant t_0$.

Following statement is generalisation of theorem 2 (Panasenko, Tonkov, 2009).

Theorem 1. *Let's assume, that there are functions $V(\sigma,x)$, $q(\sigma,z)$ and $L(z)$ such that $V(\sigma,x)$ is Lyapunov's definitely positive function concerning the set $\mathfrak{M}(\sigma)$ and for all $(\sigma,x)\in\Sigma\times\mathbb{R}^n_+$ the next inequalities are true:*

$$V^o_{\max}(\sigma,x) \leqslant q\big(\sigma, V(\sigma,x)\big), \qquad \sup_{\sigma\in\Sigma, w\in W} V\big(\sigma, x+g(x,w)\big) \leqslant \inf_{\sigma\in\Sigma} L\big(V(\sigma,x)\big). \tag{7}$$

Then, if trivial solution of equation (6) *are stable on Lyapunov* (*in classical sense*), *then the set $\mathfrak{M}(\sigma)$ is positively invariant concerning system* (2).

Proof. Let $\varphi(t)=\varphi(t,\sigma,x_0)$ is the solution of control system (2) defined on some interval $[t_0,\tau)$ and satisfying to initial condition $\varphi(t_0,\sigma,x_0)=x_0$. We will show, that for any $\varepsilon\in(0,r)$ will be such $\delta\in(0,\varepsilon)$, that for any $t_0\in\mathbb{R}$ and any solution $\varphi(t)$ from a condition $(t_0,\varphi(t_0))\in\mathfrak{M}^\delta(\sigma)$ follows that $(t,\varphi(t))\in\mathfrak{M}^\varepsilon(\sigma)$ for all $t\geqslant 0$. If the set $\mathfrak{M}(\sigma)$ possesses given property, it is called *uniformly stable on Lyapunov concerning system* (2) (Panasenko, Tonkov, 2009).

We will choose $\varepsilon\in(0,r)$ and designate

$$\alpha = \alpha(\varepsilon) = \inf_{(t,x)}\big\{V(h^t\sigma,x)\colon (t,x)\in \operatorname{fr}\mathfrak{M}^\varepsilon(\sigma)\big\}.$$

As Lyapunov's function $V(\sigma,x)$ definitely positive, then $\alpha>0$. Trivial solution of equation (6) is stable (in Lyapunov' sense), therefore for given α it is possible to find such $\delta_0\in(0,\alpha)$, that for any solution $z(t,\sigma,z_0)$ of equation (6) from an inequality $z_0<\delta_0$ follows that $z(t,\sigma,z_0)<\alpha$ for any $t\geqslant t_0$. Owing to continuity of function $x\to V(\sigma,x)$ there exists such $\delta=\delta(\delta_0)\in(0,\varepsilon)$ that $V(h^{t_0}\sigma,x)<\delta_0$ for all $x\in N^{\delta}(h^{t_0}\sigma)$.

Let's consider function $v(t,\sigma)=V\big(h^t\sigma,\varphi(t)\big)$, which is owing to Rademacher's theorem and differentiated at almost all $t\in[t_0,\tau)$. In points of differentiability of function $v(t,\sigma)$ the inequality $\dot v(t,\sigma)\leqslant V^o_{\max}\big(h^t\sigma,\varphi(t)\big)$ is executed (Panasenko, Tonkov, 2009). Therefore, from (7) we have for all $t\in[0,\tau)$ the inequality $\dot v(t,\sigma)\leqslant q(h^t\sigma,v(t,\sigma))$, which is true as $\varphi(t)\in\mathbb{R}^n_+$ (last inclusion follows from a condition of quasi-positivity of functions $f(h^t\sigma,x,u)$ and $g(x,w)$). We will designate through $z(t,\sigma,z_0)$ the solution of the equation (6), satisfying to the initial condition $z_0=v(t_0,\sigma)$.

Owing to Chaplygin's theorem of differential inequalities the inequality $v(t,\sigma)\leqslant z(t,\sigma)$ is true for all $t\in[t_0,\min\{\tau,\tau_p\})$, where $\tau_p=\tau_p(\sigma)$ is least moment of jump of the system (6), satisfying to condition $\tau_p>t_0$. Further if $\tau>\tau_p$ then from second inequality (7) it follows that for any $\sigma\in\Sigma$ and $w\in W$ the next relations are truthful:

$$
\begin{aligned}
v(\tau_p+0,\sigma)=V\big(h^{\tau_p+0}\sigma,\varphi(\tau_p+0)\big)=V\big(h^{\tau_p+0}\sigma,\varphi(\tau_p)+g(\varphi(\tau_p),w)\big)\leqslant\\
\leqslant L\big(V(h^{\tau_p}\sigma,\varphi(\tau_p))\big)=L(v(\tau_p,\sigma)).
\end{aligned}
$$

From the equality

$$z(\tau_p+0,\sigma,z_0)=z(\tau_p,\sigma,z_0)+l(z(\tau_p,\sigma,z_0))=L(z(\tau_p,\sigma,z_0))$$

we get the inequality $v(\tau_p+0,\sigma)\leqslant z(\tau_p+0,\sigma,z_0)$. Continuing similar reasoning, it is possible to show that the inequality $0\leqslant v(t,\sigma)\leqslant z(t,\sigma,z_0)$ is true for all $t\in[t_0,\tau)$.

Let's assume, that there will be time moment t^*, for which $(t^*,\varphi(t^*))\in\operatorname{fr}\mathfrak{M}^{\varepsilon}(\sigma)$. As $z_0=v(t_0,\sigma)=V(h^{t_0}\sigma,x)<\delta_0$, we receive the contradiction:

$$\alpha\leqslant v(t^*,\sigma)\leqslant z(t^*,\sigma,z_0)<\alpha.$$

Thus, the solution of system leaving at $t=t_0$ from set $M^{\delta}(\sigma)$, remains in set $\mathfrak{M}^{\varepsilon}(\sigma)$ for all $t\geqslant t_0$; therefore the given solution is infinitely continued to the right. If the closed set $\mathfrak{M}(\sigma)$ is uniformly stable (in Lyapunov' sense) concerning system (2), then it is positively invariant (Panasenko, Tonkov, 2009).

Definition 6 (Panasenko, Tonkov, 2008, 2009). The set $\mathfrak{M}(\sigma)$ is called *weakly positively invariant* (concerning system (2)) if for any initial point $(t_0,x_0)\in\mathfrak{M}(\sigma)$ there *exists* a solution $\varphi(t,\sigma,x_0)$ of systems (2) with initial condition $\varphi(t_0,\sigma,x_0)=x_0$ satisfying to inclusion $(t,\varphi(t,\sigma,x_0))\in\mathfrak{M}(\sigma)$ for all $t\geqslant t_0$.

Theorem 2. *Let's assume, that there are functions $V(\sigma,x)$, $q(\sigma,z)$ and $L(z)$ such that $V(\sigma,x)$ is definitely positive Lyapunov's function concerning set $\mathfrak{M}(\sigma)$ and for all $(\sigma,x)\in\Sigma\times\mathbb{R}^n_+$ are true inequalities*

$$V^o_{\min}(\sigma,x)\leqslant q\big(\sigma,V(x)\big),\quad \sup_{\sigma\in\Sigma}\min_{w\in W}V\big(\sigma,x+g(x,w)\big)\leqslant\inf_{\sigma\in\Sigma}L\big(V(\sigma,x)\big).$$

Then, if the trivial solution of the equation (6) *is stable on Lyapunov, then the set $\mathfrak{M}(\sigma)$ is weakly positively invariant concerning system* (2).

Proof. Owing to A. F. Filippov's theorem (Blagodatskikh, Filippov, p. 213, 1985) and theorem 2 of publication (Rodina, Tonkov, 2009) there is a solution $\varphi(t)=\varphi(t,\sigma,x_0)$ of control system (2), satisfying to initial condition $\varphi(t_0,\sigma,x_0)=x_0$, such that function $v(t,\sigma)=V\big(h^t\sigma,\varphi(t)\big)$ satisfies to an inequality $\dot v(t,\sigma)\leqslant q(h^t\sigma,v(t,\sigma))$. The further proof is similar to the proof of the theorem 1. $\square$

Asymptotical stability and extinction of population

In the following statement we receive conditions when population size $\varphi(t,\sigma,x)$, defined by system (2), approaches to given set $\mathfrak{M}(\sigma)$. In that specific case when the set $\mathfrak{M}(\sigma)$ does not depend on parameter σ and looks like $\mathfrak{M}=\{(t,x):t\geqslant 0,x=0\}$, we receive conditions of extinction of population.

Theorem 3. *Let's assume, that there are functions $V(\sigma,x)$, $q(\sigma,z)$ and $L(z)$ such that $V(\sigma,x)$ is Lyapunov's definitely positive function concerning set $\mathfrak{M}(\sigma)$ and inequalities* (7) *are satisfied for all $(\sigma,x)\in\Sigma\times\mathbb{R}^n_+$. If $\lim\limits_{t\to\infty}z(t,\sigma,z_0)=0$, then for any solution $\varphi(t,\sigma,x_0)$ of system* (2), *satisfying to initial condition $\varphi(t_0,\sigma,x_0)=x_0\in\mathbb{R}^n_+$, where $V(h^{t_0}\sigma,x_0)\leqslant z_0$, takes place equality $\lim\limits_{t\to\infty}\varrho\big(\varphi(t,\sigma,x_0),M(h^t\sigma)\big)=0$.*

Proof. Owing to the theorem 1 set $\mathfrak{M}(\sigma)$ is positively invariant concerning system (2). Let $\varphi(t)=\varphi(t,\sigma,x_0)$ is a solution of system (2), beginning in a point $(t_0,x_0)\in\mathfrak{M}^r(\sigma)$ and defined for all $t\geqslant t_0$. From the proof of theorem 1 follows that function $v(t,\sigma)=V\big(h^t\sigma,\varphi(t)\big)$ is defined at all $t\geqslant t_0$ and satisfies to an inequality of $0\leqslant v(t,\sigma)\leqslant z(t,\sigma,z_0)$, where $z(t,\sigma,z_0)$ is the solution of the equation (6) with initial condition $z_0\geqslant v(t_0,\sigma)=V(h^{t_0}\sigma,x_0)$. Thus,

$$\lim_{t\to\infty}v(t,\sigma)=\lim_{t\to\infty}V\big(h^t\sigma,\varphi(t)\big)=\lim_{t\to\infty}z(t,\sigma,z_0)=0.$$

We will show that $\lim\limits_{t\to\infty}\varrho\big(\varphi(t),M(h^t\sigma)\big)=0$. Let's assume, that if it not so, then there is a constant $\varepsilon\in(0,r)$ and sequence $\{t_i\}_{i=1}^{\infty}$ such that $t_i\to\infty$ and $\varrho\big(\varphi(t_i),M(h^{t_i}\sigma)\big)\geqslant\varepsilon$. It means, that $(t_i, \varphi(t_i)) \notin \mathfrak{M}^{\varepsilon}(\sigma)$ and as function V is definitely positive, then it will be such $\delta > 0$, that $V\big(h^{t_i}\sigma, \varphi(t_i)\big) \geqslant \delta$. We received the contradiction with a condition $\lim\limits_{t\to\infty} V\big(h^t\sigma, \varphi(t)\big) = 0$.

As a corollary of theorem 3 we received conditions of extinction of the population which dynamics is given by control system (2).

Theorem 4. *Let's assume, that there is Lyapunov's definitely positive function $V(\sigma, x) = V(x)$ concerning set $\mathfrak{M} = \{(t, x) : t \geqslant 0, x = 0\}$ and functions $q(\sigma, z)$, $L(z)$ such that for all $(\sigma, x) \in \Sigma \times \mathbb{R}^n_+$ are true inequalities*

$$V^o_{\max}(\sigma, x) \leqslant q\big(\sigma, V(x)\big), \quad \max_{w\in W} V\big(x + g(x, w)\big) \leqslant L\big(V(x)\big).$$

Then, if $\lim\limits_{t\to\infty} z(t, \sigma, z_0) = 0$ and $V(x_0) \leqslant z_0$, for any solution $\varphi(t, \sigma, x_0)$ of systems (2) take place equality $\lim\limits_{t\to\infty} |\varphi(t, \sigma, x_0)| = 0$.

Let's notice, that if equality $\lim\limits_{t\to\infty} z(t, \sigma, z_0) = 0$ is true not for all, but for almost all $\sigma \in \Sigma$, then $\lim\limits_{t\to\infty} |\varphi(t, \sigma, x)| = 0$ also for almost all $\sigma \in \Sigma$, that is population degenerates with probability one. Conditions of equality to zero of limit $\lim\limits_{t\to\infty} z(t, \sigma, z_0)$ (which are executed for all $\sigma \in \Sigma$ or with probability one) are received in work (Rodina, 2014).

In the following statement we received conditions of existence of control, leading population to extinction (we considered population of harmful insects, viruses or bacteria).

Theorem 5. *Let's assume, that there is Lyapunov's definitely positive function $V(\sigma, x) = V(x)$ concerning set $\mathfrak{M} = \{(t, x) : t \geqslant 0, x = 0\}$ and functions $q(\sigma, z)$, $L(z)$ such that for all $(\sigma, x) \in \Sigma \times \mathbb{R}^n_+$ are true inequalities*

$$V^o_{\min}(\sigma, x) \leqslant q\big(\sigma, V(x)\big), \quad \min_{w\in W} V\big(x + g(x, w)\big) \leqslant L\big(V(x)\big). \tag{8}$$

Then, if $\lim\limits_{t\to\infty} z(t, \sigma, z_0) = 0$ and $V(x_0) \leqslant z_0$, there exists the solution $\varphi(t, \sigma, x_0)$ of system (2), satisfying to the initial condition $\varphi(t_0, \sigma, x_0) = x_0$ such that $\lim\limits_{t\to\infty} |\varphi(t, \sigma, x_0)| = 0$.

The proof follows from the proof of the previous theorem and results of works (Blagodatskikh, Filippov, 1985; Panasenko, Tonkov, 2009).

Control of number of bisexual population, leading to its extinction with probability one

For research of dynamics of isolated population we will use probabilistic discrete-continuous model which is generalization of deterministic model described in work (Nedorezov, Utyupin, 2003). Let $x_1(t)$ is a number of male and $x_2(t)$ is a number of female individuals, that at the time moment t satisfy to following control system:

$$\begin{aligned}
\dot{x_1} &= -(a_1+u_1)x_1 - b_1x_1(x_1+\gamma x_2), \quad t \neq \tau_k(\sigma), \\
\dot{x_2} &= -(a_2+u_2)x_2 - b_2x_2(x_1+\gamma x_2), \quad t \neq \tau_k(\sigma), \\
\Delta x_1\Big|_{t=\tau_k(\sigma)} &= w_1D - x_1, \quad \Delta x_2\Big|_{t=\tau_k(\sigma)} = w_2D - x_2,
\end{aligned} \tag{9}$$

where a_1, a_2 are coefficients of natural death of male and female individuals, b_1, b_2 are coefficients of self-regulation. Coefficient γ reflects inadequacy of "contribution"individuals of various sexes in the self-control process, all factors specified above are positive. Further, u_1, u_2 are controls influencing on factors of natural death and satisfying restrictions $u_1 \in [u_1^1, u_1^2]$, $u_2 \in [u_2^1, u_2^2]$, where $u_1^1 + a_1 > 0$, $u_2^1 + a_2 > 0$. Variable

$$D = \min\{x_2(\tau_k), \varepsilon x_1(\tau_k)\}$$

is equal to number of impregnated females at moment $\tau_k(\sigma)$, where ε is "activity factor"of males which reflects not only their potential possibilities, but also character of interaction of individuals of various sexes. In particular, if all individuals strictly break into couples, $\varepsilon = 1$. Variables w_1, w_2 are equal to an average of descendants of male and female sexes accordingly, generated by one impregnated female; we will consider the case when these sizes can be control so, that $w_1 \in [w_1^1, w_1^2]$, $w_2 \in [w_2^1, w_2^2]$, where $w_1^1 > 0$, $w_2^1 > 0$. Not reducing a generality of analysis we put $\varepsilon = 1$.

We assume, that the moments τ_k depend on various natural conditions, therefore lengths of intervals between the moments of appearance of new generation $\theta_k = \tau_k - \tau_{k-1}$, $k = 2, 3, \ldots$ are independent random variables with distribution $G(t)$, which is concentrated on a segment $[\alpha, \beta]$, where $0 < \alpha < \beta < \infty$. In publication (Nedorezov, Utyupin, 2003) it is noticed that analysis of dynamics of number of bisexual population is important problem both with theoretical, and from a practical position. For example, there are various control methods of number of harmful species of insects, focused on creation of some unbalance in sexual structure of population that promotes decrease in speed of its reproduction and quite often leads to degeneration.

As Lyapunov's function we take function $V(x_1, x_2) = x_1 + x_2$. We will find

$$V^o_{\min}(x) = -(a_1+u_1^2)x_1 - b_1x_1(x_1+\gamma x_2) - (a_2+u_2^2)x_2 - b_2x_2(x_1+\gamma x_2),$$

then for all $x \in \mathbb{R}^2_+$ takes place following inequality $V^o_{\min}(x) \leqslant -V(x)\big(aV(x)+b\big)$, where $a = \min\Big(b_1, \gamma b_2, \dfrac{\gamma b_1 + b_2}{2}\Big) > 0$, $b = \min(a_1+u_1^2, a_2+u_2^2) > 0$. Further,

$$\min_{w\in W} V(x+g(x,w)) = \min_{w_1,w_2}(w_1+w_2)D = (w_1^1+w_2^1)\min(x_1,x_2) \leqslant \frac{w_1^1+w_2^1}{2}\, V(x).$$

Thus, inequalities (8) are true for functions

$$V(x) = x_1 + x_2, \;\; q(\sigma, z) = -z(az + b), \;\; L(z) = cz, \quad \text{where} \quad c = \frac{w_1^1 + w_2^1}{2}.$$

For given functions we will compile the differential equation

$$\dot{z} = -z(az + b), \quad t \neq \tau_k(\sigma), \quad \Delta z\big|_{t=\tau_k(\sigma)} = (c-1)z, \tag{10}$$

that parameterized by metric dynamical system $(\Sigma, \mathfrak{A}, \nu, h^t)$. It is simple to consider, that for $c \in (0, 1]$ the size of the population, given by the equation (10) (hence, by system (9)), asymptotically tends to zero, therefore we will assume further, that $c > 1$.

Let's assume that random variables θ_k, $k = 2, 3, \ldots$ have the distribution concentrated on a segment $[\alpha, \beta]$. On each interval (τ_k, τ_{k+1}) the solution of equation (10) is function

$$\eta(t, z_k) = \frac{bz_k}{az_k(e^{b(t-\tau_k)} - 1) + be^{b(t-\tau_k)}},$$

where $z_k = \eta(\tau_k+) = \lim\limits_{t \to \tau_k + 0} \eta(t)$. Let's construct the function

$$H(t, z) = L(\eta(t, z)) = \frac{bcz}{az(e^{bt} - 1) + be^{bt}}$$

and find solutions of equation $H(t, z) = z$: $z_1 = 0$, $z_2 = \dfrac{b(c - e^{bt})}{a(e^{bt} - 1)}$, where $z_2 > 0$ for $t < \dfrac{\ln c}{b}$. Thus, from results of publication (Rodina, 2014) follows, that if $\alpha \geqslant \dfrac{\ln c}{b}$, then for any $\sigma \in \Sigma$ and any $z_0 \geqslant 0$ is executed equality $\lim\limits_{t \to \infty} z(t, \sigma, z_0) = 0$, where $z(t, \sigma, z_0)$ is a solution of the equation (10), satisfying to initial condition $z(t_0, \sigma, z_0) = z_0$. Owing to the theorem 5 for any $x_0 \in \mathbb{R}_+^2$ there will be a solution $\varphi(t, \sigma, x_0)$ of system (9), satisfying to initial condition $\varphi(t_0, \sigma, x_0) = x_0$, for which equality $\lim\limits_{t \to \infty} |\varphi(t, \sigma, x_0)| = 0$ is true. It means, that if $\alpha \geqslant \dfrac{\ln c}{b}$, then exists the control, resulting population (9) to degeneration.

Let's find conditions of existence a control resulting population (9) to extinction with probability one. For this purpose we will use the statement of work (Rodina, 2014):

Theorem. *Let's assume that there exists a measurable set* $I \subseteq [\alpha, \beta]$ *such that* $\mu(I) > 1/2$, *for everyone* $t \in I$ *the equation* $H(t, z) = z$ *has no positive solutions and is true the inequality*

$$\sup_{t \in I, z > 0} \frac{H(t, z)}{z} \cdot \sup_{t \in [\alpha, \beta], z > 0} \frac{H(t, z)}{z} < 1.$$

Then the population, which dynamics is given by the model (10), *degenerates with probability one, that is there is a set* $\Sigma_0 \subseteq \Sigma$ *such that* $\mu(\Sigma_0) = 1$ *and* $\lim\limits_{t \to \infty} z(t, \sigma, z_0) = 0$ *for all* $\sigma \in \Sigma_0$ *and any* $z_0 \geqslant 0$.

Let $I=[\alpha_0,\beta_0]\subseteq[\alpha,\beta]$. It is simple to receive, that the theorem statement is true, if $\alpha_0>\dfrac{\ln c}{b}$, $\alpha+\alpha_0>2\dfrac{\ln c}{b}$ and $\mu(I)>1/2$.

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