About a non-parametric continuous-discrete model of parasite-host system dynamics

Neklyudova V.L.[1], Nedorezov L.V.[2]

[1]Siberian State Academy of Geodesy
E-mail: neklyudova@ssga.ru

[2]University of Nova Gorica, Slovenia
E-mail: l.v.nedorezov@gmail.com

Key words: parasite-host system dynamics, continuous-discrete mathematical model

Abstract. In paper continuous-discrete model of parasite-host system dynamics is analyzed. Within the framework of model it is assumed that appearance of individuals of new generations of both populations is realized at fixed time moments $t_k = hk$, $t_0 = 0$, $k = 1,2,\ldots$, $h = const > 0$; it means that several processes are compressed together: producing of eggs by hosts, attack of eggs by parasites (with respective transformation of host's eggs into parasite's eggs), staying of hosts and parasites in phase "egg", and appearance of individuals of next generations. It is also assumed that death process of individuals has a continuous nature, but developments of both populations are realized independently between fixed time moments. Dynamic regimes of model are analyzed.

1. Introduction

Dynamics of parasite-host system is analyzed in a lot of various publications (Nicholson, Bailey, 1935; Kostitzin, 1937; Maynard Smith, 1968, 1974; Beddington, Free, Lawton, 1975; Isaev et al., 1984, 2001, 2009; Kot, 2001; Brauer, Castillo-Chavez, 2001; Nedorezov, 1986, 1997, 2012 and many others). Dynamics of this system can be described with the help of ordinary differential equations or recurrence equations, and in most cases there are no differences between parasite-host and predator-prey systems. In particular, it depends on the level of generality of description of a process of interaction of species. But in the case of using of continuous-discrete models (Kostitzin, 1937; Nedorezov, 1986, 1997, 2012; Nedorezov, Nedorezova, 1994, 1995; Nedorezov, Utyupin, 2011) we have to divide between these qualitatively different cases.

As it was demonstrated in our previous publications (Nedorezov, 2012; Nedorezov, Utyupin, 2011) rather realistic description of predator-prey system dynamics leads to necessity in introduction of one more variable into model which corresponds to volume of consumed food or level of saturation of predators. Productivity process in population of predators may depend on average of level of saturation during a certain time interval. Note the last cannot be taken into account within the framework of every model of predator-prey system dynamics with discrete time.

In current publication we analyze continuous-discrete model of parasite-host system dynamics with non-overlapping generations. It is also assumed that appearance of individuals of new generations are observed at nearest fixed time moments for both populations that can be described as a jump of model trajectory.

2. Model

Let $x(t)$ and $y(t)$ be the numbers of hosts and parasites respectively at time t. We'll assume that developments of both populations are realized in one-year generations at fixed time moments $t_k = hk$, $k = 0,1,2\ldots$ Respectively, appearance of individuals of new generation correlates with death of all individuals of previous generation. Denote as $x(t_k - 0)$ number of hosts survived to moment t_k and $y(t_k - 0)$ number of parasites at the same time moment.

We will also assume that at moments t_k survived hosts produce eggs with a certain rate. Let Y be an average number of eggs produced by one survived host. Thus number of produced eggs E_k will be determined by the following relation:

$$E_k = Yx(t_k - 0).$$

Below we'll analyze a situation when $Y \equiv const > 1$. It is obvious if the inverse inequality is truthful both populations extinct for all initial values of populations. Let's also assume that at moments t_k parasites attack host's eggs, and denote as P a quota of infected eggs. Amount $Yx(t_k - 0)P$ is equal to total number of infected eggs. Additional assumption is following: every infected egg was attacked by only parasite, and every infected egg can be transformed into one parasite at moment t_k.

Let's consider first variant when P is determined on number of parasites survived to moment t_k, $P = P(y(t_k - 0))$. It means that within the framework of model parasites have "unlimited potential": when number of parasites is fixed increase of number of hosts leads to linear increasing of infected eggs on $x(t_k - 0)$. In general case amount of P depends on $x(t_k - 0)$ and $y(t_k - 0)$, and additionally parasite's potential must be limited.

Following the biological sense of function P next conditions must be truthful:

1) increase of number of parasites $y(t_k - 0)$ leads to monotonic increase of function P and $P(\infty) = 1$;
2) if $y(t_k - 0) = 0$ function P is equal to zero too.

In simplest case P can be presented as fractional-linear function:

$$P = P(y(t_k - 0)) = \frac{y(t_k - 0)}{q + y(t_k - 0)}.$$

Amount of parameter $q = const > 0$ depends on conditions for parasites in their search of host's eggs. Increase of value of this parameter q leads to decrease of number of attacked hosts.

We'll also assume that sojourn time of individuals in phase "egg" and "infected egg" is rather small, and there is no necessity to consider respective processes in model. Thus, we'll assume that transformation of eggs into adults of new generation is realized at moment t_k. Finally, we'll assume that at moments t_k we have following relations for adults for hosts and parasites:

$$x_k = Yx(t_k - 0)(1 - P(y(t_k - 0))),$$
$$y_k = Yx(t_k - 0)P(y(t_k - 0)). \quad (1)$$

It is naturally to assume that between selected time moments t_k there are no interactions between hosts and parasites: this is one of basic differences between parasite-host and predator-prey systems. Moreover, on time intervals $[t_k, t_{k+1})$ we can observe monotonic decreasing of population's sizes which we'll describe with following differential equations:

$$\frac{dx}{dt} = -xR_1(x),$$
$$\frac{dy}{dt} = -yR_2(y), \quad (2)$$

Conditions (1) are initial values for system (2) for every interval $[t_k, t_{k+1})$. Functions $R_1(x)$ and $R_2(y)$ describe death processes in populations and correspond to standard conditions:

$$R_1(0) = R_1^0 > 0,\ R_1(\infty) = \infty,\ \frac{dR_1}{dx} > 0,$$
$$R_2(0) = R_2^0 > 0,\ R_2(\infty) = \infty,\ \frac{dR_2}{dy} > 0. \quad (3)$$

Let

$$\psi_1(s) = \int \frac{ds}{sR_1(s)},\ \psi_2(s) = \int \frac{ds}{sR_2(s)}. \quad (4)$$

As follows from conditions (3), functions (4) are monotonic increasing functions with negative second derivatives. Thus, for both functions (4) we have the inverse ones ψ_1^{-1} and ψ_2^{-1} which are monotonic increasing functions with positive second derivatives. Use of conditions (3) allows obtaining the following relations from system (2):

$$x(t_k - 0) = \psi_1^{-1}(C_1 - t_k),\ y(t_k - 0) = \psi_2^{-1}(C_2 - t_k).$$

Constants C_1 and C_2 can be calculated with initial conditions:

$$C_1 = t_{k-1} + \psi_1(x_{k-1}),\ C_2 = t_{k-1} + \psi_2(y_{k-1}).$$

Thus, continuous-discrete model (1)-(2) can be transformed into system of the following recurrence equations:

$$x_k = Y\psi_1^{-1}(\psi_1(x_{k-1}) - h)(1 - P(\psi_2^{-1}(\psi_2(y_{k-1}) - h)),$$
$$y_k = Y\psi_1^{-1}(\psi_1(x_{k-1}) - h)P(\psi_2^{-1}(\psi_2(y_{k-1}) - h). \quad (5)$$

3. Properties of model

1. For non-negative and finite initial values of populations x_0 and y_0 trajectories of model are non-negative and bounded.

2. Origin is stationary state of system (5). The following inequality is necessary and sufficient condition for global stability of origin:

$$Ye^{-R_1^0 h} < 1. \quad (6)$$

3. If in (6) the inverse inequality is truthful system (5) has one stationary state $(D,0)$ on x line. Stationary state $(D,0)$ is stable if and only if the following inequality is truthful:

$$DP'(0)e^{-R_2^0 h} < 0. \quad (7)$$

4. If inverse inequalities in (6) and (7) are truthful there is unique stationary state of system (5) in positive part of phase plane. This stationary state is stable.

4. Conclusion

Analyzed non-parametric model of parasite-host system dynamics showed if parasites have unlimited possibilities in infection of host's eggs in model we have regimes with one non-trivial stationary state (in positive part of phase plane). Respectively, we cannot obtain outbreak regimes which can be characterized by certain number of equilibriums in positive part (Isaev et al., 1984, 2001, 2009). Moreover, non-trivial stationary state is stable. It gives a certain guarantee that following dynamic regimes can be observed within the framework of considered model: extinction of both populations, extinction of parasites with stabilization of hosts at non-zero level, and stabilization of both populations at non-zero levels.

5. References

Beddington J.R., Free C.A., Lawton J.H. 1975. Dynamic and complexity in predator-prey models framed in difference equations// Nature 255: 58-60.

Brauer F., Castillo-Chavez C. 2001. Mathematical Models in Population Biology and Epidemiology. N.Y.: Springer-Verlag, 417 pp.

Isaev A.S., Khlebopros R.G., Nedorezov L.V., Kondakov Yu.P., Kiselev V.V. 1984. Forest Insect Population Dynamics. Nauka, Novosibirsk, 224 pp.

Isaev A.S., Khlebopros R.G., Nedorezov L.V., Kondakov Yu.P., Kiselev V.V., Soukhovolsky V.G. 2001. Population Dynamics of Forest Insects. Nauka, Moscow, 347 pp.

Isaev A.S., Khlebopros R.G., Kondakov Yu.P., Kiselev V.V., Nedorezov L.V., Soukhovolsky V.G. 2009. Forest Insect Population Dynamics// Euroasian Entomological Journal 8(2): 3-115.

Kostitzin V.A. 1937. La Biologie Mathematique. A.Colin, Paris, 236 pp.

Kot M. 2001. Elements of Mathematical Ecology. Cambridge University Press, Cambridge, 453 pp.

Maynard Smith J. 1968. Mathematical Ideas in Biology. Cambridge University Press, Cambridge, 152 pp.

Maynard Smith J. 1974. Models in Ecology. Cambridge University Press, Cambridge, 145 pp.

Nedorezov L.V. 1986. Modeling of Forest Insect Outbreaks. Nauka, Novosibirsk, 125 pp.

Nedorezov L.V. 1997. Course of Lectures on Ecological Modeling. Siberian Chronograph, Novosibirsk, 161 pp.

Nedorezov L.V. 2012. Chaos and Order in Population Dynamics: Modeling, Analysis, Forecast. LAP Lambert Academic Publishing, Saarbrucken, 352 pp.

Nedorezov L.V., Nedorezova B.N. 1994. Modifications of Moran-Ricker Models of Isolated Population Dynamics. J. of General Biology 55(4-5): 514-521.

Nedorezov L.V., Nedorezova B.N. 1995. Correlation between Models of Population Dynamics in Continuous and Discrete Time. Ecological Modelling 82: 93-97.

Nedorezov L.V., Utyupin Yu.V. 2011. Continuous-Discrete Models of Population Dynamics: An Analytical Overview. Vip. 95. State Public Scientific-Technical Library of Siberian Branch of Russian Academy of Sciences, Novosibirsk ,234 pp.

Nicholson A.J., Bailey V.A. 1935. The balance of animal populations// Proc. Zool. Soc. London: 551-598.